Integration

Danang Mursita

Integration

- The Indefinite Integral
- The Definite Integral
- The Fundamental Theorem of Calculus
- Application of Integration : Area between two curves

The Indefinite Integral

- Definition : A function F(x) is called an antiderivative of a function f(x) if the derivative of F(x) is f(X) or F '(x) = f(x)
- Examples :
  1. $F(x) = x^2 + 3 \rightarrow f(x) = 2x$
  2. $F(x) = x^2 - 10 \rightarrow f(x) = 2x$
  3. $F(x) = x^2 + 15 \rightarrow f(x) = 2x$
- If F '(x) = f(x) then the functions of the form F(x) + C are antiderivative of f(x)
- The process for finding antiderivative is called **antidifferentiation** or **integration**.
- Notation : $\int f(x)\,dx = F(x) + C$

Integration Formula

$$1.\ \frac{d}{dx}(\sin x) = \cos x \Leftrightarrow \int \cos x\, dx = \sin x + C$$

2....

3....etc

$$a.\ \frac{d}{dx}(x) = 1 \Leftrightarrow \int dx = x + C$$

$$b.\ \frac{d}{dx}\left(\frac{x^{r+1}}{r+1}\right) = \ldots$$

$$\int x^r dx = \frac{x^{r+1}}{r+1} + C\left[r \neq -1\right]$$

Problems

1) $\int \frac{\sin x}{\cos^2 x} dx$

2) $\int \frac{\cos x}{\sin^2 x} dx$

3) $\int \sqrt{x}\, dx$

4) $\int \sqrt[3]{x^2}\, dx$

5) $\int \frac{1}{\sqrt{x^3}} dx$

6) $\int \sec x(\tan x + \cos x) dx$

7) $\int \frac{\sin 2x}{\cos x} dx$

8) $\int dx$

9) $\int (x+2)\sqrt[3]{x^2}\, dx$

10) $\int \frac{x^2 - 3}{\sqrt{x^3}} dx$

The Properties of Indefinite Integral

$$1)\int cf(x)\,dx = c\int f(x)\,dx$$

$$2)\int \left[f(x)+g(x)\right]dx = \int f(x)\,dx + \int g(x)\,dx$$

Examples :

$$1.\int x^2(x+1)\,dx$$

$$2.\int \sqrt{x}(x-1)^2\,dx$$

$$3.\int \frac{x^2\sin x+\sqrt{x}}{x^2}\,dx$$

Integration by Substitution

- Let y = (fog)(x). The derivative of y with respect to x is y ' = (fog)'(x) = f '(g(x)) g'(x) [The Chain rule]
- If u = g(x) then the formula of integration is

$$\int f(u)u'\,dx = \int f(u)\frac{du}{dx}\,dx = \int f(u)\,du = F(u) + C = F\big(g(x)\big) + C$$

Problems

1) $\int 2x\sqrt{x^2+1}\,dx$

2) $\int \cos\left(1-x^2\right)x\,dx$

3) $\int \frac{x+1}{\sqrt{x^2+2x-1}}\,dx$

4) $\int \sin x \cos^2 x\,dx$

5) $\int \cos x \sin^2 x\,dx$

6) $\int (1-2x)^5\,dx$

7) $\int \frac{\sin 2x}{(5+\cos 2x)^3}\,dx$

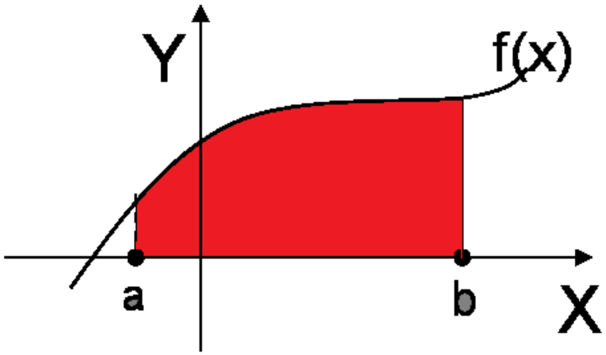
8) $\int \frac{x^2}{\sqrt{x^3+1}}\,dx$

9) $\int \frac{\sin\left(1/x\right)}{x^2}\,dx$

10) $\int x^2 \sec^2\left(x^3\right)dx$

Problem : The Definite Integral

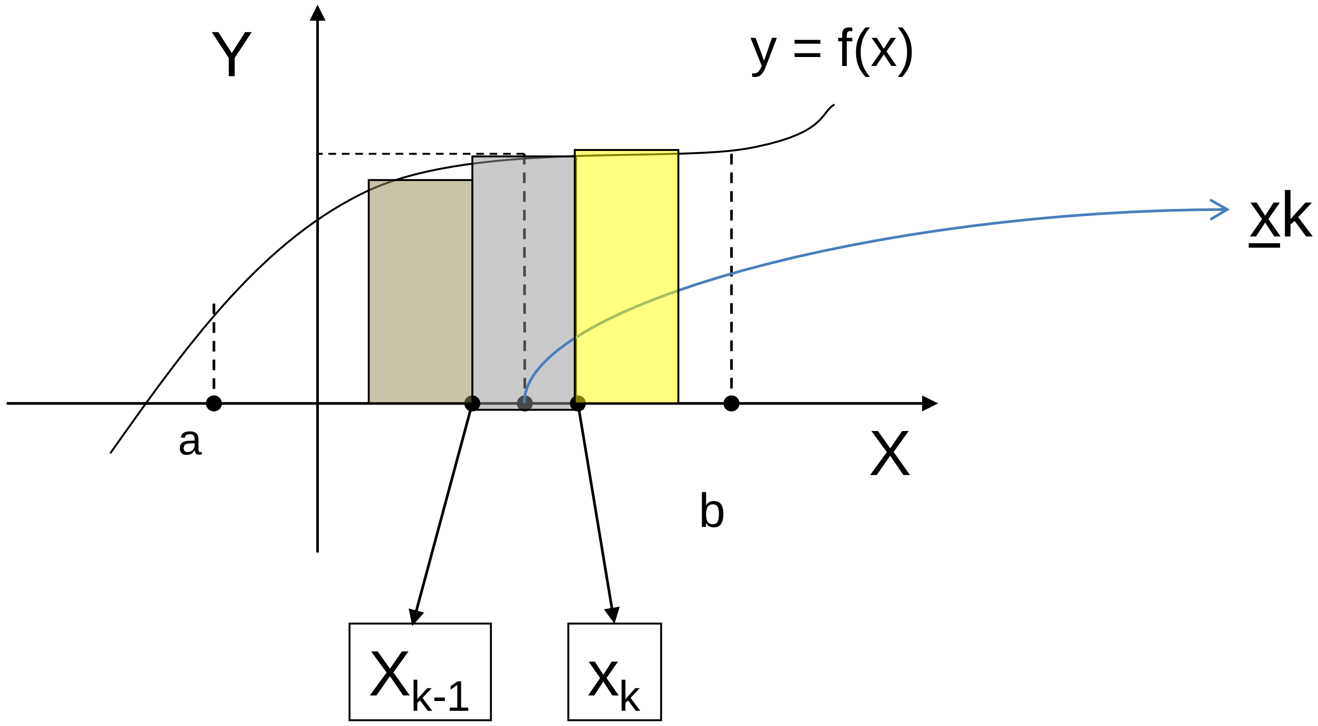
- How to find the area of a region D bounded below by the x-axis, on the sides by the lines x = a and x = b , and above by a curve y = f(x), where f(x) is continuous on [a,b] and f(x) ≥ 0 for all x in (a,b)
- D = { (x,y) |) }



Steps : The Definite Integral

1. Divide [a,b] into n subintervals by choosing points x1, x2, …,xn such that a < x1 < x2 < …< xn < b. These points are said to form a partition of [a,b]. Let $\Delta x1$, $\Delta x2$, …, Δxn are the length of the partition.
2. Choose the any point $\underline{x}1, \underline{x}2, \ldots, \underline{x}n$ in subinterval and we have a Riemann sum :
   $$f(\underline{x}1)\,\Delta x1 + f(\underline{x}2)\,\Delta x2 + \ldots + f(\underline{x}n)\,\Delta xn = \sum_{k=1}^{n} f(\underline{x}k)\Delta xk$$
3. Increase n (n → ∞) in such a way the length of the partition approaches zero ($\Delta xk \to 0$) and form the limit
   $$\lim_{\Delta xk \to 0} \sum_{k=1}^{n} f(\underline{x}k)\Delta xk = \lim_{n\to\infty} \sum_{k=1}^{n} f(\underline{x}k)\Delta xk$$

Illustration : The Definite Integral



Movie : The Definite Integral

Rieman_b.exe

poligon.exe

Definition : The Definite Integral

- If a function f(x) is defined on [a,b], then the **Definite Integral** of f(x) from a to b is defined to be (provided the limit exits)

$$\int_a^b f(x)dx = \lim_{\Delta xk \to 0} \sum_{k=1}^{n} f(\underline{x}k)\Delta xk = \lim_{n \to \infty} \sum_{k=1}^{n} f(\underline{x}k)\Delta xk$$

- If the limit exits then the function f(x) is said **integrable** on [a,b]
- The values a and b are called respectively **lower** and **upper limits of integration** and f(x) is called the **Integrand**.

The First Fundamental Theorem

- If f(x) is continuous on [a,b] and F(x) is antiderivative of f(x) on [a,b] then

$$\int_a^b f(x)\,dx = F(x)\Big|_a^b = F(b) - F(a)$$

Examples :

$$(1)\ \int_1^2 x\,dx$$

$$(2)\ \int_0^{\pi/2} \cos x\,dx$$

The Properties of Definite Integral

$$(1)\ \int_a^b [p\,f(x) + q\,g(x)]\,dx = p\int_a^b f(x)\,dx + q\int_a^b g(x)\,dx$$

$$(2)\ \int_a^a f(x)\,dx = 0$$

$$(3)\ \int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

$$(4)\ \int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

$$(5)\ \int_a^b (fog)(x)\,g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du$$

Problems

$$(1)\int_{1}^{3}\frac{1}{x^2}\,dx$$

$$(2)\int_{4}^{9}2x\sqrt{x}\,dx$$

$$(3)\int_{1}^{4}\left[\frac{3}{\sqrt{x}}-5\sqrt{x}-x^{-3/2}\right]dx$$

$$(4)\int_{\pi/6}^{\pi/2}\left(\frac{2}{\sin^2 t}+t\right)dt$$

$$(5)\int_{0}^{2}\left|2x-3\right|dx$$

$$(6)\int_{-1}^{2}\sqrt{2+|x|}\,dx$$

$$(7)\int_{-1}^{2}x\sqrt{8-x^2}\,dx$$

$$(8)\int_{1}^{3}\frac{x+2}{\sqrt{x^2+4x+7}}\,dx$$

$$(9)\int_{0}^{\pi/4}\sqrt{\tan t}\,\sec^2 t\,dt$$

$$(10)\int_{0}^{\pi/4}\frac{\cos 2t}{\sqrt{7-3\sin 2t}}\,dt$$

The Second Fundamental Theorem

- Let f(x) is continuous on open interval I and a $\in$ I. If G(x) is defined by

$$G(x) = \int_a^x f(t)\,dt$$

Then G'(x) = f(x) at each point x in interval I

$$G(x) = \int_{u(x)}^{v(x)} f(t)\,dt \quad \Leftrightarrow \quad G'(x) = f\big(v(x)\big)v'(x) - f\big(u(x)\big)u'(x)$$

Problems

$$(1)\ G(x) = \int_{2}^{x} \sqrt{3t^2 + 1}\, dt \rightarrow G'(2) \text{ and } G''(2)$$

$$(2)\ H(x) = \int_{0}^{x} \frac{\cos t}{t^2 + 3}\, dt \rightarrow H'(0) \text{ and } H''(0)$$

$$(3)\ F(x) = \int_{0}^{x} \frac{t - 3}{t^2 + 7}\, dt \quad [-\infty < x < \infty]$$

(a) find x where F(x) attains its minimum value

(b) find open interval which F(x) is only increasing

$$(4)\ F(x) = \int_{x^2}^{x^3} \sin^2 t\, dt \rightarrow F'(x) \text{ and } F''(x)$$

Area of Region (1)

- Let D is region that bounded below by the x-axis, on the sides by the lines x = a and x = b , and above by a curve y = f(x), where f(x) is continuous on [a,b] and f(x) ≥ 0 for all x in (a,b) or D = { (x,y) | a ≤ x ≤ b, 0 ≤ y ≤ f(x) }
- Then the area of D is

$$A = \int_a^b f(x)\,dx$$

Area of Region (2)

- Let D is region that bounded below by the x-axis, on the sides by the lines x = a and x = b , and above by a curve y = f(x), where f(x) is continuous on [a,b] and f(x) ≤ 0 for all x in (a,b) or D = { (x,y) | a ≤ x ≤ b, f(x) ≤ y ≤ 0 }
- Then the area of D is

$$A = -\int_a^b f(x)\,dx$$

Example # 1

- Find the area of region that bounded by the curve $y = \sin x$, x-axis, and on the sides by the lines $x = 0$ and $x = 3\pi/2$

Area of Region (3)

- Let D is region that bounded below by the line y = c , on the sides by the y-axis and a curve x = g(y), and above by the line y = d, where g(y) is continuous on [c,d] and $g(y) \geq 0$ for all y in (c,d) or $D = \{ (x,y) \mid 0 \leq x \leq g(y), c \leq y \leq d \}$
- Then the area of D is

$$A = \int_c^d g(y)\,dy$$

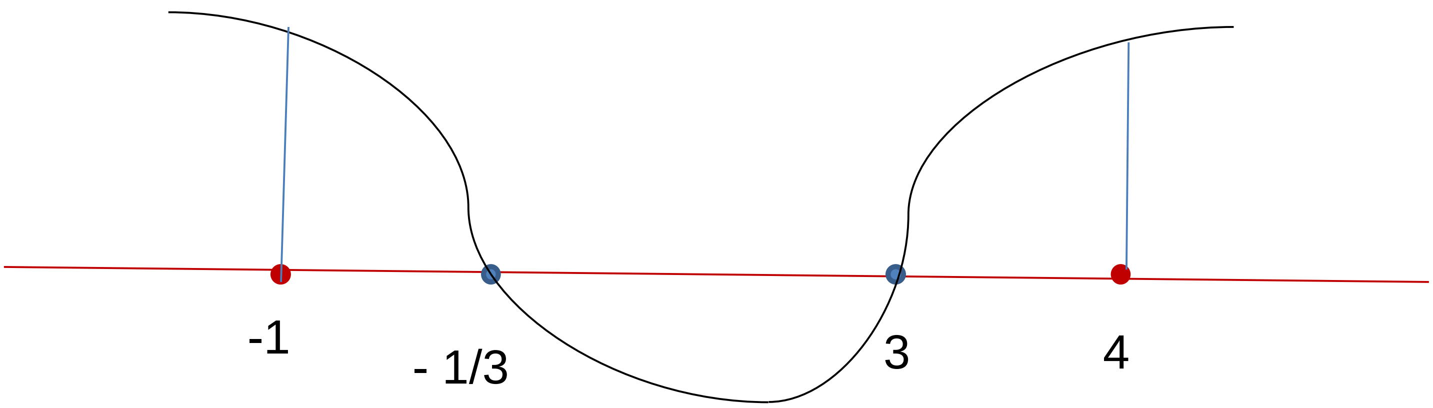
Area of Region (4)

- Let D is region that bounded below by the line $y = c$, on the sides by the y-axis and a curve $x = g(y)$, and above by the line $y = d$, where $g(y)$ is continuous on $[c,d]$ and $g(y) \le 0$ for all y in (c,d) or $D = \{ (x,y) \mid g(y) \le x \le 0, c \le y \le d \}$
- Then the area of D is

$$A = -\int_{c}^{d} g(y)\,dy$$

Example # 2

- Find the area of region that bounded by $g(y) = y^3 - 2y^2 - 3y$, y-axis and on the sides by the lines $y = -1$ and $y = 4$.



Area between two curves (1)

- Let D is region that bounded below by a curve y = g(x), on the sides by the lines x = a and x = b , and above by a curve y = f(x), where g(x) and f(x) are continuous on [a,b] or D = { (x,y) | a ≤ x ≤ b, g(x) ≤ y ≤ f(x) }
- Then the area of D is

$$A = \int_a^b \left[f(x) - g(x) \right] dx$$

Area between two curves (2)

- Let D is region that bounded below by the line y = c , on the sides by a curve x = g(y) and a curve x = f(y), and above by the line y = d, where g(y) and f(y) are continuous on [c,d] and f(y) ≥ g(y) for all y in (c,d) or D = { (x,y) | g(y) ≤ x ≤ f(y), c ≤ y ≤ d }
- Then the area of D is

$$A = \int_{c}^{d} \left[f(y) - g(y) \right] dy$$

Examples

1. Find the area of the region enclosed by the curves $y = x^2$ and $y = 4x$ by integrating
   a) With respect to x
   b) With respect to y
2. Sketch the region enclosed by the curves and find its area by any method
   a) $y = x^2 + 4$ and $x + y = 6$
   b) $y = x^3$, $y = - x$ and $y = 8$

Problems

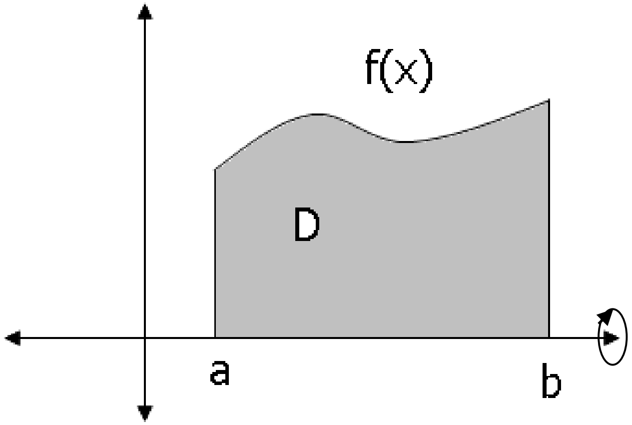
Sketch the region and find its area by any methods

1. $y = x^3 - 4x$, $y = 0$, $x = 0$, $x = 2$
2. $x = y^2 - 4y$, $x = 0$, $y = 0$, $y = 4$
3. $x^2 = y$, $x = y - 2$
4. $y^2 = -x$, $y = x - 6$, $y = -1$, $y = 4$
5. $y = x^3 - 2x^2$, $y = 2x^2 - 3x$, $x = 0$, $x = 3$

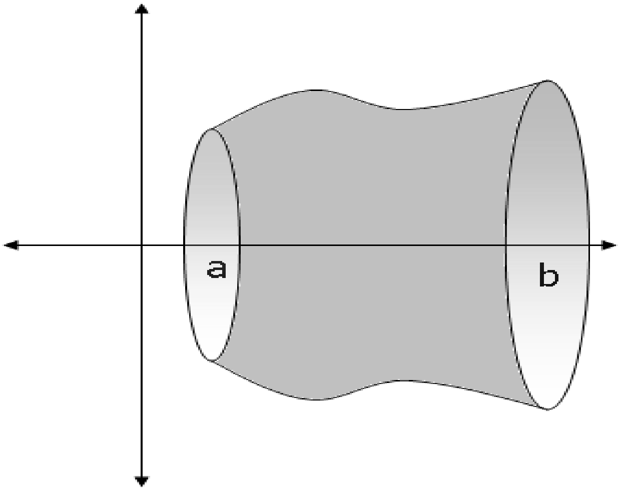
Menghitung volume benda putar

Metoda Cakram

a. Daerah $D=\{(x,y)\,|\,a\le x\le b\,,\,0\le y\le f(x)\}$ diputar terhadap sumbu x

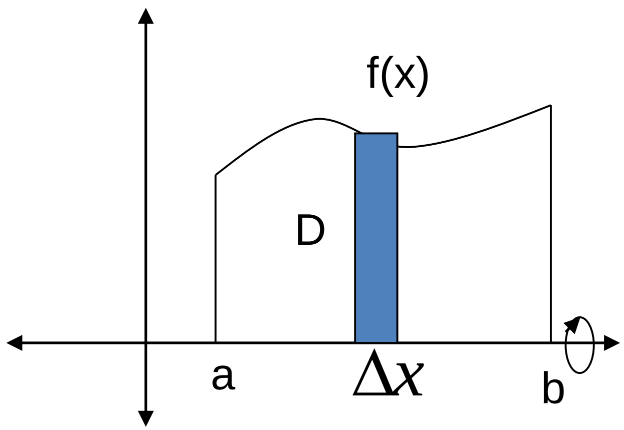


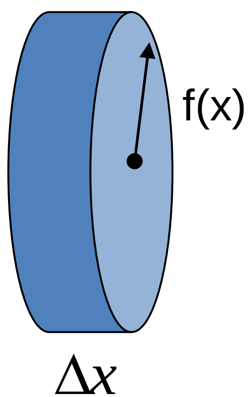
Daerah D



Benda putar

Untuk menghitung volume benda putar gunakan pendekatan Iris , hampiri, jumlahkan dan ambil limitnya.





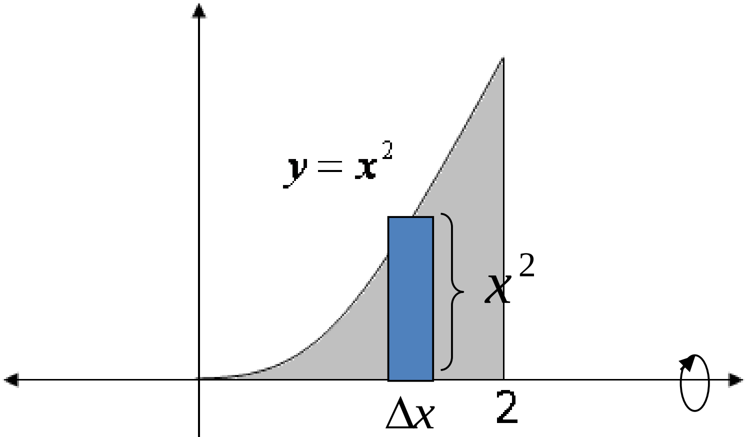
Jika irisan berbentuk persegi panjang dengan tinggi f(x) dan alas Δx diputar terhadap sumbu x akan diperoleh suatu cakram lingkaran dengan tebal Δx dan jari-jari f(x).

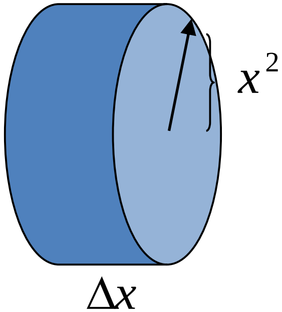
sehingga

$$\Delta V \approx \pi f^2(x)\Delta x$$

$$V = \pi \int_a^b f^2(x)\,dx$$

Contoh: Tentukan volume benda putar yang terjadi jika daerah D yang dibatasi oleh $y = x^2$, sumbu x, dan garis x=2 diputar terhadap sumbu *x*





Jika irisan diputar terhadap sumbu x akan diperoleh cakram dengan jari-jari x^2 dan tebal Δx

Sehingga

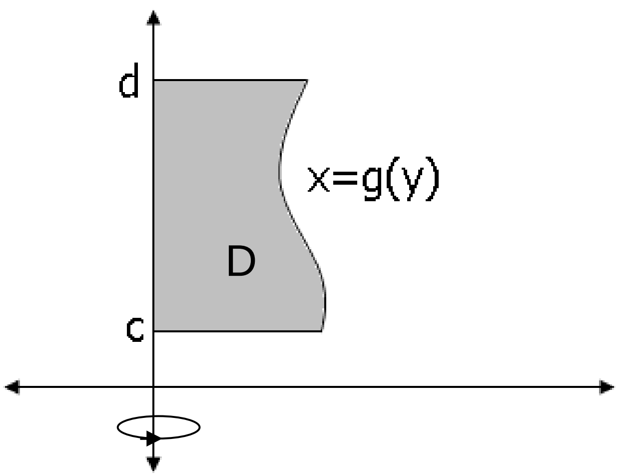
$$\Delta V \approx \pi (x^2)^2 \Delta x = \pi x^4 \Delta x$$

Volume benda putar

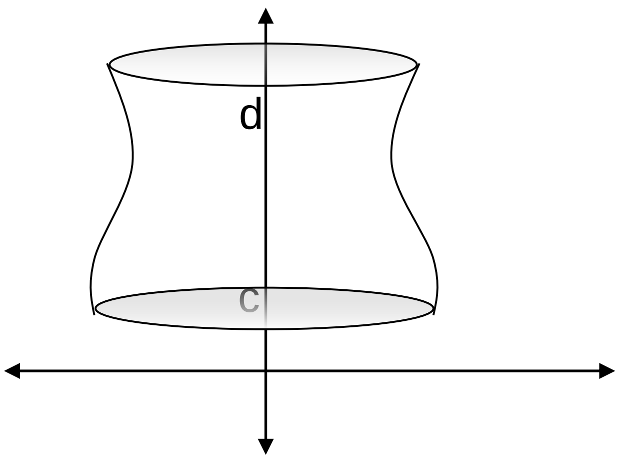
$$V = \pi \int_0^2 x^4 dx = \frac{\pi}{5} x^5 \Big|_0^2 = \frac{32}{5}\pi$$

b. Daerah $D = \{(x, y) \mid c \le y \le d \;,\; 0 \le x \le g(y)\}$

diputar terhadap sumbu y

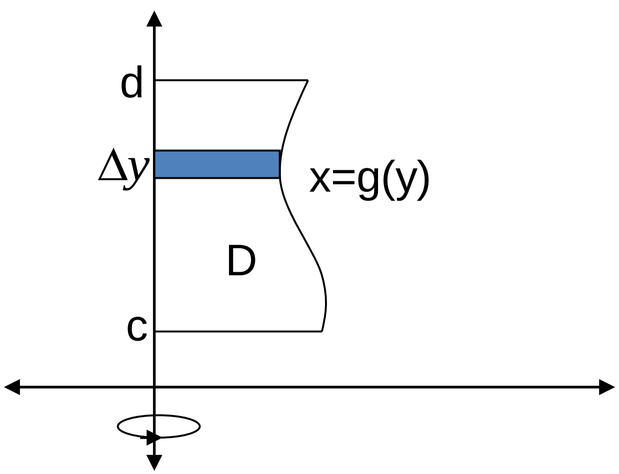


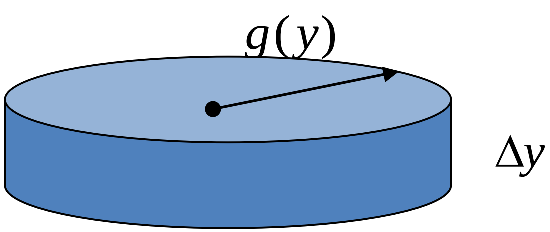
Daerah D



Benda putar

Untuk menghitung volume benda putar gunakan pendekatan Iris , hampiri, jumlahkan dan ambil limitnya.





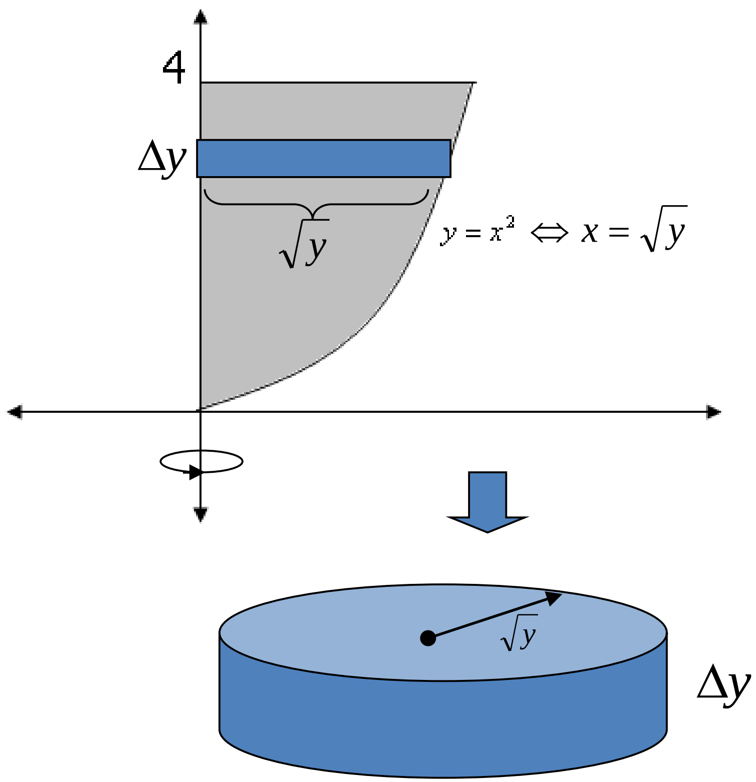
Jika irisan berbentuk persegi panjang dengan tinggi g(y) dan alas Δy diputar terhadap sumbu y akan diperoleh suatu cakram lingkaran dengan tebal Δy dan Jari-jari g(y).

sehingga

$$\Delta V \approx \pi\, g^2(y)\, \Delta y$$

$$V = \pi \int_c^d g^2(y)\, dy$$

Contoh : Tentukan volume benda putar yang terjadi jika daerah yang dibatasi oleh $y = x^2$ garis y = 4, dan sumbu y diputar terhadap sumbu y



Jika irisan dengan tinggi $\sqrt{y}$ dan tebal Δy diputar terhadap sumbu y akan diperoleh cakram dengan jari-jari $\sqrt{y}$ dan tebal Δy

Sehingga

$$\Delta V = \pi(\sqrt{y})^2 \Delta y = \pi\, y\, \Delta y$$

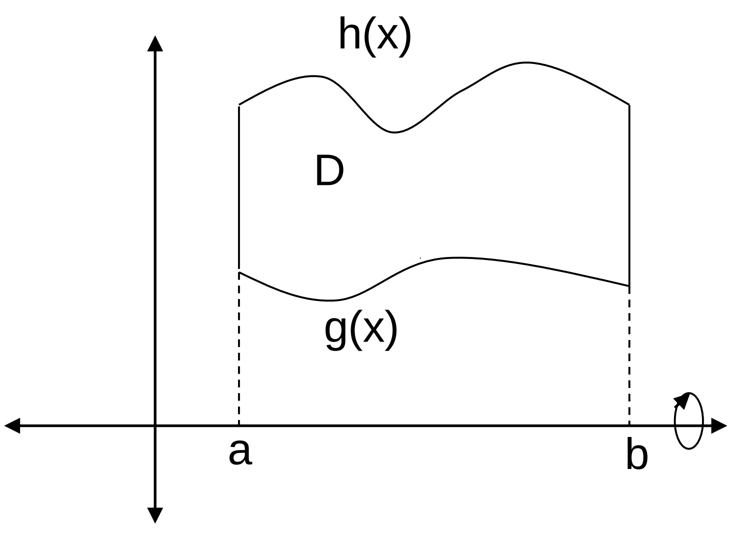
Volume benda putar

$$V = \pi \int_0^4 y\,dy = \frac{\pi}{2} y^2 \Big|_0^4 = 8\pi$$

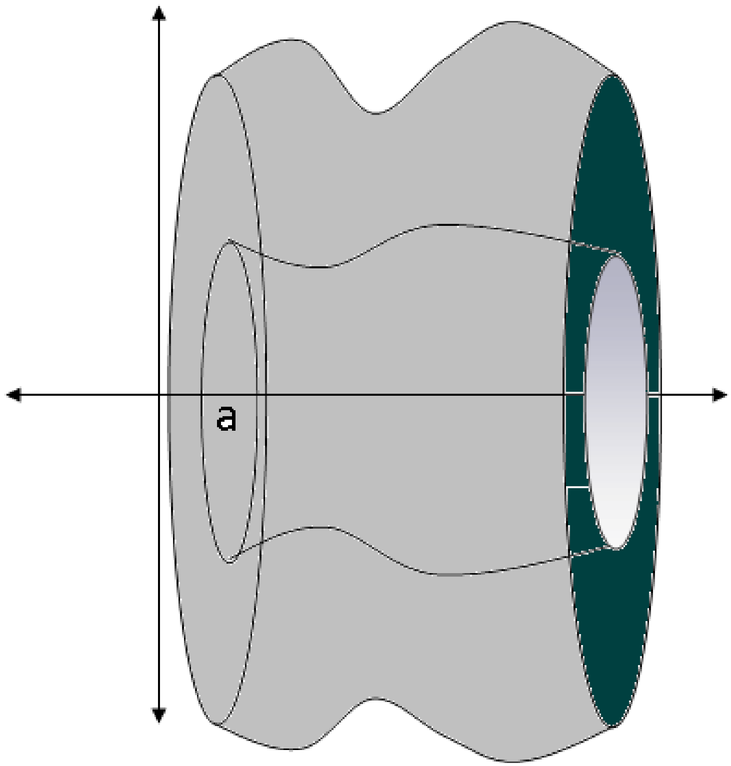
7.2.2 Metoda Cincin

a. Daerah $D = \{(x, y) \mid a \le x \le b \text{ , } g(x) \le y \le h(x)\}$

diputar terhadap sumbu x

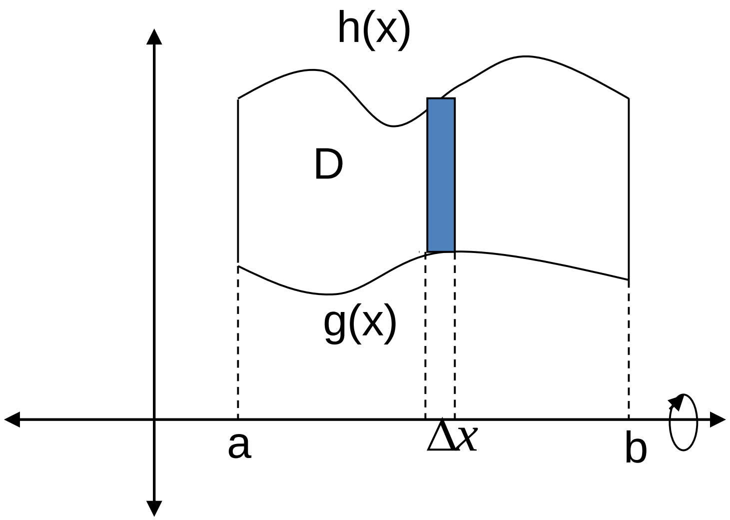


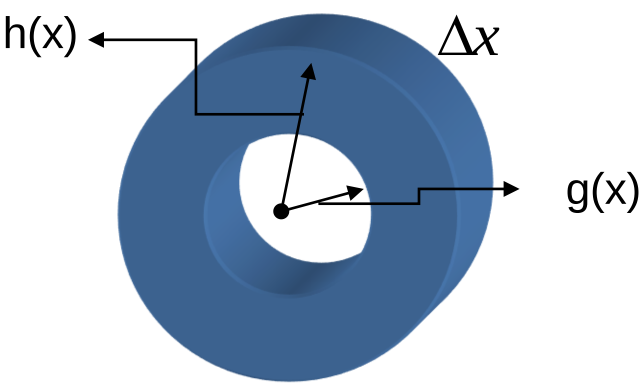
Daerah D



Benda putar

Untuk menghitung volume benda putar gunakan pendekatan Iris , hampiri, jumlahkan dan ambil limitnya.





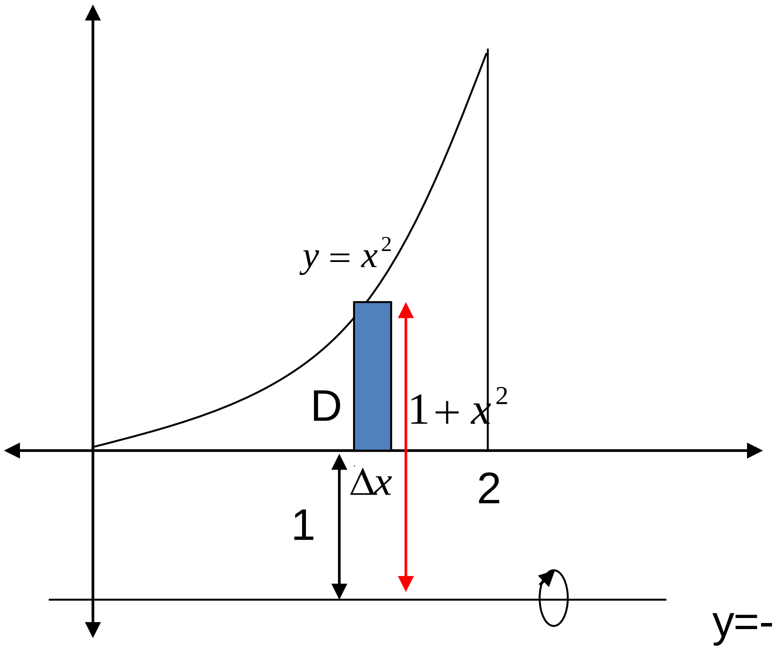
Jika irisan berbentuk persegi panjang dengan tinggi h(x)-g(x) dan alas Δx diputar terhadap sumbu x akan diperoleh suatu cincin dengan tebal Δx dan jari –jari luar h(x) dan jari-jari dalam g(x).

sehingga

$$\Delta V \approx \pi(h^2(x) - g^2(x))\Delta x$$

$$V = \pi\int_a^b (h^2(x) - g^2(x))dx$$

Contoh: Tentukan volume benda putar yang terjadi jika daerah D yang dibatasi oleh $y = x^2$, sumbu x, dan garis x=2 diputar terhadap garis y=-1



Jika irisan diputar terhadap garis y=1
Akan diperoleh suatu cincin dengan
Jari-jari dalam 1 dan jari-jari luar $1 + x^2$

Sehingga

$$\Delta V = \pi((x^2+1)^2 - 1^2)\Delta x$$

$$= \pi(x^4 + 2x^2 + 1 - 1)\Delta x$$

$$= \pi(x^4 + 2x^2)\Delta x$$

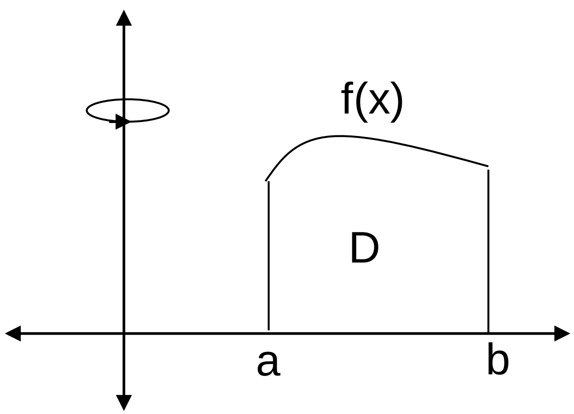
Volume benda putar :

$$V = \pi \int_0^2 x^4 + 2x^2 dx = \pi(\tfrac{1}{5}x^5 + \tfrac{2}{3}x^3 \Big|_0^2) = \pi(\tfrac{32}{5} + \tfrac{16}{3}) = \tfrac{186}{15}\pi$$

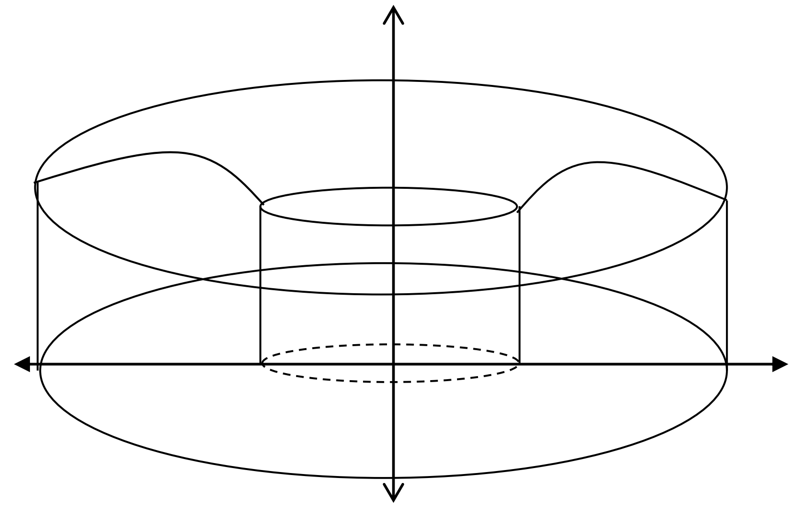
Metoda Kulit Tabung

Diketahui $D = \{(x, y) \mid a \le x \le b \,,\, 0 \le y \le f(x)\}$

Jika D diputar terhadap sumbu y diperoleh benda putar



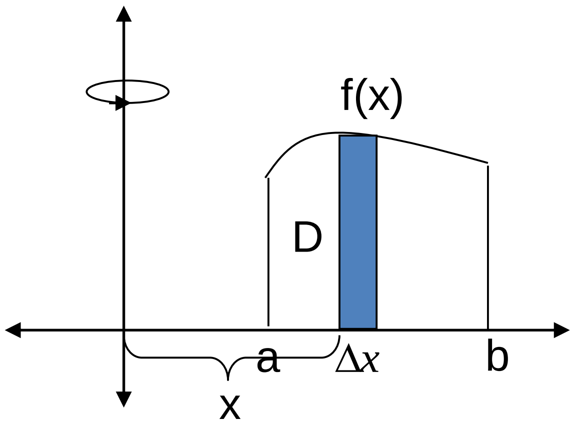
Daerah D



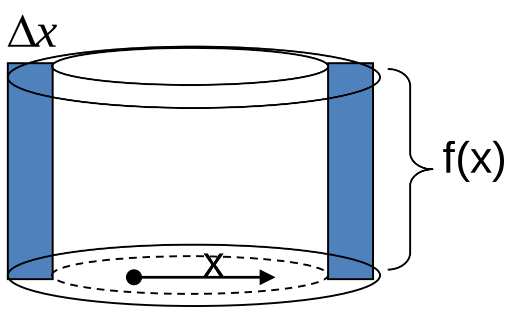
Benda putar

Volume benda putar ?

Untuk menghitung volume benda putar gunakan pendekatan Iris , hampiri, jumlahkan dan ambil limitnya.



Jika irisan berbentuk persegi panjang dengan tinggi f(x) dan alas Δx serta berjarak x dari sumbu y diputar terhadap sumbu y akan diperoleh suatu kulit tabung dengan tinggi f(x), jari-jari x, dan tebal Δx

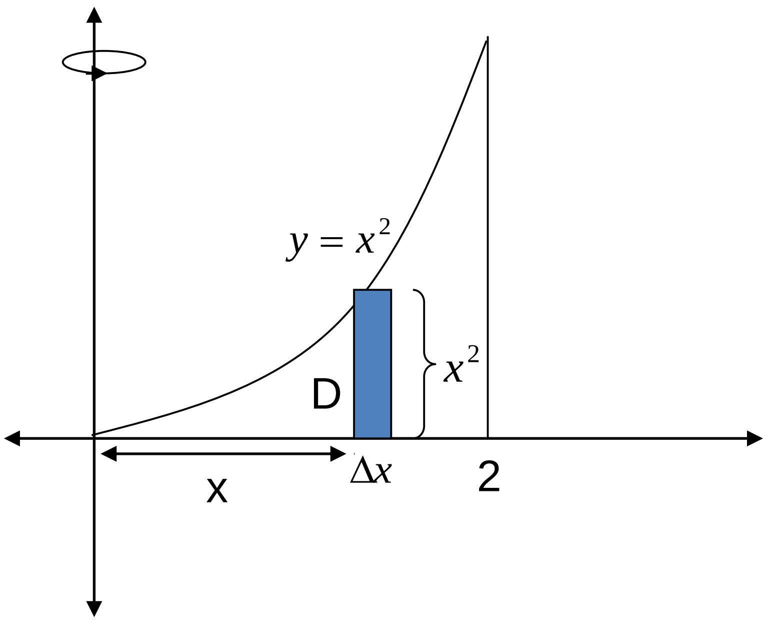


sehingga

$$\Delta V \approx 2\pi\, x\, f(x)\, \Delta x$$

$$V = 2\pi \int_a^b x f(x)\, dx$$

Contoh: Tentukan volume benda putar yang terjadi jika daerah D yang dibatasi oleh $y = x^2$, sumbu x, dan garis x=2 diputar terhadap sumbu y



Jika irisan dengan tinggi x^2 ,tebal Δx dan berjarak x dari sumbu y diputar terhadap sumbu y akan diperoleh kulit tabung dengan tinggi x^2, tebal Δx dan jari jari x

Sehingga

$$\Delta V = 2\pi \, x \, x^2 \Delta x = 2\pi \, x^3 \, \Delta x$$

Volume benda putar

$$V = 2\pi \int_0^2 x^3 dx = \frac{\pi}{2} x^4 \Big|_0^2 = 8\pi$$

Catatan :

-Metoda cakram/cincin

Irisan dibuat tegak lurus terhadap sumbu putar

- Metoda kulit tabung

Irisan dibuat sejajar dengan sumbu putar

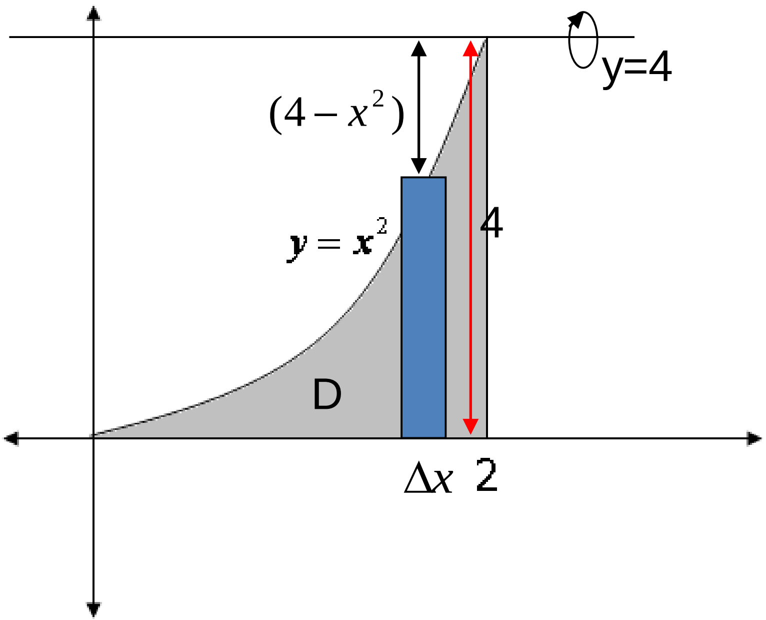
Jika daerah dan sumbu putarnya sama maka perhitungan dengan menggunakan metoda cakram/cincin dan metoda kulit tabung akan menghasilkan hasil yang sama

Contoh Tentukan benda putar yang terjadi jika daerah D yang dibatasi Oleh parabola $y = x^2$,garis x = 2, dan sumbu x diputar terhadap

a. Garis y = 4
b. Garis x = 3

a. Sumbu putar y = 4

(i) Metoda cincin



Jika irisan diputar terhadap garis y=4 akan diperoleh cincin dengan

Jari-jari dalam = $r_d = (4 - x^2)$

Jari-jari luar = $r_l = 4$

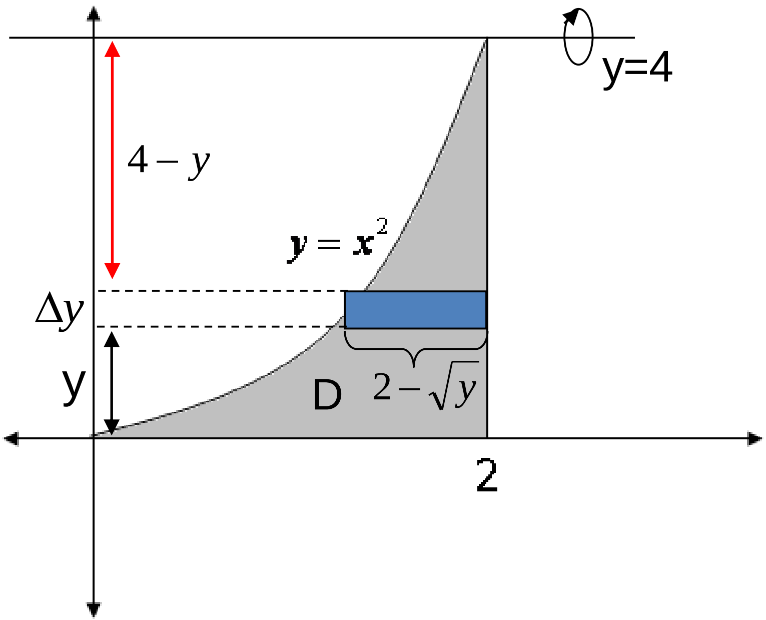
Sehingga

$$\Delta V \approx \pi((4)^2 - (4 - x^2)^2)\Delta x$$
$$= \pi(8x^2 - x^4)\Delta x$$

Volume benda putar

$$V = \pi \int_0^2 (8x^2 - x^4)dx = \pi(\tfrac{8}{3}x^3 - \tfrac{1}{5}x^5)\Big|_0^2 = \pi(\tfrac{64}{3} - \tfrac{32}{5}) = \tfrac{224}{15}\pi$$

(ii) Metoda kulit tabung



Jika irisan diputar terhadap garis y=4 akan diperoleh kulit tabung dengan

Jari-jari = r = $4-y$

Tinggi = h = $2-\sqrt{y}$

Tebal = Δy

Sehingga

$$\Delta V \approx 2\pi(4-y)(2-\sqrt{y})\Delta y$$

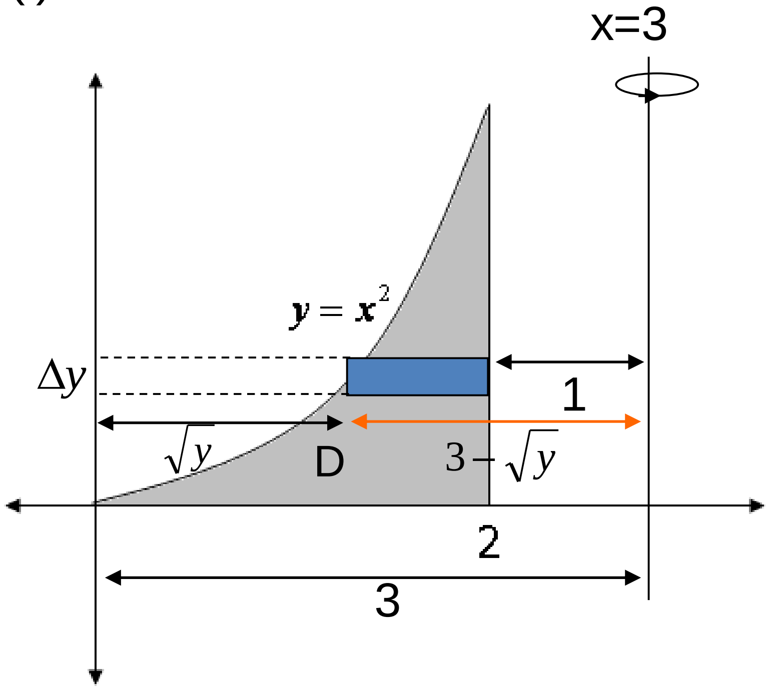
$$= 2\pi(8-4\sqrt{y}-2y+y\sqrt{y})\Delta y$$

Volume benda putar

$$V = 2\pi\int_0^4 (8-4\sqrt{y}-2y+y\sqrt{y})dy = 2\pi(8y-\tfrac{8}{3}y^{3/2}-y^2+\tfrac{2}{5}y^{5/2})\Big|_0^4 = \frac{224}{15}\pi$$

b. Sumbu putar x=3

(i) Metoda cincin



Jika irisan diputar terhadap garis x=3 diperoleh cincin dengan

Jari-jari dalam = $r_d = 1$

Jari-jari luar = $r_l = 3 - \sqrt{y}$

Sehingga

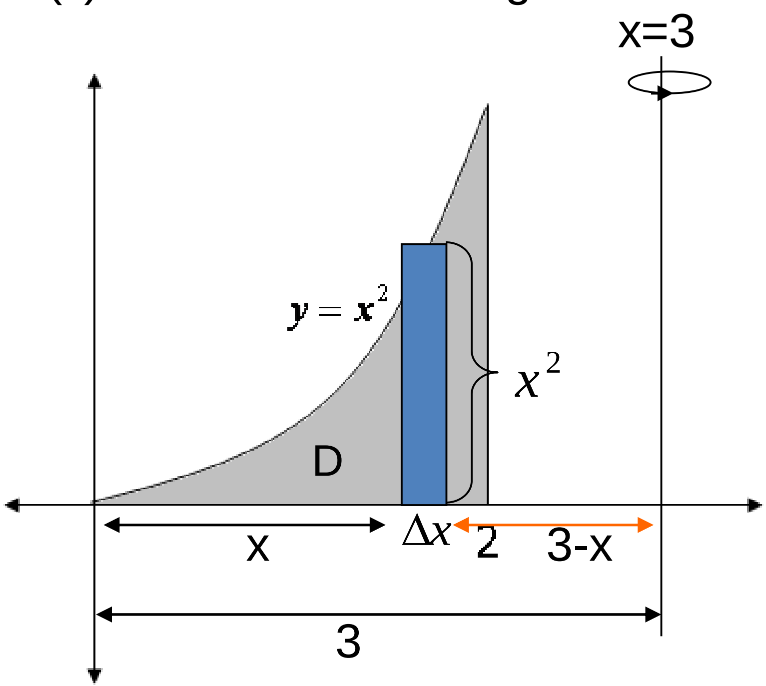
$$\Delta V \approx \pi((3-\sqrt{y})^2 - (1)^2)\Delta y$$

$$= \pi(8 - 6\sqrt{y} + y)\Delta y$$

Volume benda putar

$$V = \pi\int_0^4 (8-6\sqrt{y}+y)dy \quad = \pi(8y - 4y^{3/2} + 8 \Big|_0^4) = 8\pi$$

(ii) Metoda kulit tabung



Jika irisan diputar terhadap garis x=3 diperoleh kulit tabung dengan

Tinggi = h = x^2

Jari-jari = r = 3-x

Tebal = Δx

Sehingga

$$\Delta V \approx 2\pi(3-x)x^2\Delta x$$
$$= 2\pi(3x^2 - x^3)\Delta x$$

Volume benda putar

$$V = 2\pi\int_0^2 (3x^2 - x^3)dx = 2\pi(x^3 - \tfrac{1}{4}x^4)\Big|_0^2 = 2\pi(8-4) = 8\pi$$

Soal Latihan

Hitung volume benda putar dari daerah yang terletak di kuadran pertama yang dibatasi oleh $y^2 = x^3$, garis y = 8 dan sumbu Y, bila diputar mengelilingi

1. Sumbu Y
2. Sumbu X
3. Garis x = 4
4. Garis y = 8


Thank You!