Techniques of Integration

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Integration By Part

$$\frac{d}{dx}\left[f(x)g(x)\right] = f(x)\frac{d\left(g(x)\right)}{dx} + g(x)\frac{d\left(f(x)\right)}{dx}$$

$$\int \frac{d}{dx}\left[f(x)g(x)\right]dx = \int f(x)\frac{d\left(g(x)\right)}{dx}dx + \int g(x)\frac{d\left(f(x)\right)}{dx}dx$$

$$f(x)g(x) = \int f(x)\frac{d\left(g(x)\right)}{dx}dx + \int g(x)\frac{d\left(f(x)\right)}{dx}dx$$

$$\int f(x)g'(x)dx = f(x)g(x) - \int g(x)f'(x)dx$$

$$\int udv = uv - \int vdu$$

$$\int_a^b udv = uv\Big|_a^b - \int_a^b vdu$$

Trick and Tips

- If the integrand has two factors ie the polynomial and the trigonometric function or transcendental function then substitute u = the polynomial
- If the integrand has a factor trigonometric function and transcendental function then substitute u = the trigonometric function or u = the transcendental function

Examples

Use the integration by part for calculating this integral

$$(1).\ \int (x-2)\cos 2x\,dx$$

$$(2).\ \int_0^1 \left(x^2+1\right)e^x\,dx$$

$$(3).\ \int e^{-x}\cos x\,dx$$

$$(4).\ \int_{-1}^1 \ln(x+2)\,dx$$

Trigonometric Integral

We calculate integral of f(x) with integrand f(x) is represented by

1. $\mathrm{Sin}^n x$, $\cos^n x$
2. $\sin^m x \cos^n x$
3. sin (ax) cos (bx),
4. sin (ax) sin(bx),
5. cos (ax) cos (bx)
6. $\tan^m x \sec^n x$,
7. $\cot^m x \csc^n x$

$\int \cos^n x \; dx \; \& \int \sin^n x \; dx$

1. Let n is odd. Then
   - Represent $\cos^n x = \cos x \cos^{n-1} x$ and $\sin^n x = \sin x \sin^{n-1} x$
   - Use $\sin^2 x + \cos^2 x = 1$, $d(\sin x) = \cos x \, dx$ and $d(\cos x) = - \sin x \, dx$
2. Let n is even. Then
   - Use $\cos 2x = 2 \cos^2 x - 1 = 1 - 2 \sin^2 x$ or
   - Represent $\cos^n x = \cos x \cos^{n-1} x$ and $\sin^n x = \sin x \sin^{n-1} x$ then use integral by part for calculating this integral

Examples

Calculate this integral

$$(1).\int \cos^3 x\,dx$$

$$(2).\int_0^{\pi/4} \sin^2(2x)\,dx$$

$$(3).\int \cos^4(3x)\,dx$$

$$(4).\int_0^{\pi/2} \sin^5 x\,dx$$

$$\int \sin^m x \cos^n x \, dx$$

1. n is odd, substitute u = sin x, identity : $\cos^2 x = 1 - \sin^2 x$
2. m is odd, substitute u = cos x, identity : $\sin^2 x = 1 - \cos^2 x$
3. m and n are even, use identity to reduce the powers on sin and cos, identity : $\sin^2 x = ½ (1 - \cos 2x)$ and $\cos^2 x = ½ (1 + \cos 2x)$

Examples

Calculate this integral

$$(1).\int \cos^5 x \sin x \, dx$$

$$(2).\int \cos^4 x \sin^3 x \, dx$$

$$(3).\int_0^{\pi/3} \sin^4(3x)\cos^3(3x)\,dx$$

$$(4).\int_0^{\pi/2} \sin^2 \frac{x}{2} \cos^2 \frac{x}{2}\,dx$$

$$\int \sin(ax)\cos(bx)dx, \int \sin(ax)\sin(bx)dx, \int \cos(ax)\cos(bx)dx$$

- Use this identity

$$\sin(ax)\cos(bx) = \frac{1}{2}\left[\sin(a+b)x + \sin(a-b)x\right]$$

$$\sin(ax)\sin(bx) = -\frac{1}{2}\left[\cos(a+b)x - \cos(a-b)x\right]$$

$$\cos(ax)\cos(bx) = \frac{1}{2}\left[\cos(a+b)x + \cos(a-b)x\right]$$

Examples

Calculate this integral

$$(1). \int \cos 3x \sin 2x \, dx$$

$$(2). \int \sin x \sin \frac{x}{2} \, dx$$

$$(3). \int_0^{\pi/6} \cos(2x)\cos(4x) \, dx$$

$$(4). \int_0^{\pi/2} \sin \frac{x}{2} \cos x \, dx$$

$\int \tan^m x \sec^n x \, dx$ & $\int \cot^m x \csc^n x \, dx$

1. $m = 1$ and $n = 0$, substitute : $d(\cos x) = -\sin x \, dx$ and $d(\sin x) = \cos x \, dx$
2. $m = 0$ and $n = 1$, substitute : $d(\sec x) = \sec x \tan x \, dx$ and $d(\csc x) = -\csc x \cot x \, dx$
3. $m > 1$ and $n = 0$ or $m = 0$ and $n > 1$, use identity : $\tan^2 x = \sec^2 x - 1$ and $\cot^2 x = \csc^2 x - 1$

Examples

Calculate this integral

$(1).\int \tan x\, dx$

$(2).\int \cot x\, dx$

$(3).\int \sec x\, dx$

$(4).\int \csc x\, dx$

$(5).\int \tan^2 x\, dx$

$(6).\int \cot^3 x\, dx$

$(7).\int \sec^3 x\, dx$

$(8).\int \csc^4 x\, dx$

$$\int \tan^m x \sec^n x \, dx$$

1. n is even, substitute $u = \tan x$, identity : $\sec^2 x = \tan^2 x + 1$
2. m is odd, substitute $u = \sec x$, identity : $\tan^2 x = \sec^2 x - 1$
3. m is even and n is odd, reduce to powers of sec

$$\int \cot^m x \csc^n x \, dx$$

1. n is even, substitute $u = \cot x$, identity : $\csc^2 x = \cot^2 x + 1$
2. m is odd, substitute $u = \csc x$, identity : $\cot^2 x = \csc^2 x - 1$
3. m is even and n is odd, reduce to powers of csc

Examples

Calculate this integral

$$(1).\int \tan^3 x \sec^4 x \, dx$$

$$(2).\int \cot^3 x \csc^3 x \, dx$$

$$(3).\int_0^{\pi/6} \sec^3 x \tan x \, dx$$

$$(4).\int_{\pi/4}^{\pi/2} \csc^3 x \cot x \, dx$$

Trigonometric Substitutions

- This method can be used for calculating integration with integrand has a factor of

(1). $\sqrt{a^2 - x^2}$

(2). $\sqrt{a^2 + x^2}$

(3). $\sqrt{x^2 - a^2}$

Trigonometric Substitutions

$$(1).\ \text{Substitute } x = a\sin t \Rightarrow$$

$$\sqrt{a^2 - x^2} = a\cos t,\ dx = a\cos t\, dt$$

$$(2).\ \text{Substitute } x = a\tan t \Rightarrow$$

$$\sqrt{a^2 + x^2} = a\sec t,\ dx = a\sec^2 t\, dt$$

$$(3).\ \text{Substitute } x = a\sec t \Rightarrow$$

$$\sqrt{x^2 - a^2} = a\tan t,\ dx = a\sec t\tan t\, dt$$

Examples

$$(1).\int \sqrt{9-x^2}\,dx$$

$$(2).\int \frac{x^2}{\sqrt{1+x^2}}\,dx$$

$$(3).\int \frac{dx}{\left(x^2-4\right)^{3/2}}$$

$$(4).\int \frac{\sqrt{1+x^2}}{x}\,dx$$

$$(1).\int_0^4 x^3\sqrt{16-x^2}\,dx$$

$$(2).\int_{\sqrt{2}}^{2} \frac{dx}{x^2\sqrt{x^2-1}}$$

$$(3).\int_0^{1/3} \frac{dx}{\left(4-9x^2\right)^2}$$

$$(4).\int_{-1/\sqrt{2}}^{1/\sqrt{2}} \left(1-2x^2\right)^{3/2}dx$$

Integrals involving $ax^2 + bx + c$

Integrals that involve $ax^2 + bx + c$ with $b \neq 0$ can be solved by

1. Substitution $u = x + b/2a$, ie :

$$ax^2 + bx + c = a\,(x^2 + b/a\;x) + c$$
$$= a\,(x^2 + b/a\;x + b^2/4a^2) + c - b^2/4a$$
$$= a\,(\,x + b/2a)^2 + c - b^2/4a$$

2. We have a simple form $au^2 + d$ and then trigonometric substitution can be applied

Examples

$$(1).\int \frac{dx}{x^2 - 6x + 13}$$

$$(2).\int \frac{x}{x^2 + 6x + 10}\,dx$$

$$(3).\int \frac{x+3}{\sqrt{x^2 + 2x + 2}}\,dx$$

$$(4).\int_1^2 \frac{dx}{\sqrt{4x - x^2}}$$

Integral of Rational Function

- The function f(x) = P(x)/ Q(x) is Rational Function, ie : P(x) and Q(x) are Polynomials.
- Integral of f(x) which is rational function with degree of numerator, P(x) less than degree of denominator, Q(x) can be solved by the possibility of factor of the denominator, Q(x) :
  - Q(x) have linear factor (repeat and non repeat factor)
  - Q(x) have non linear factor (repeat and non repeat factor)

The Linear Factor

1. $Q(x) = (x - q1)(x - q2)\ldots(x - qn)$

$$f(x) = \frac{A1}{x - q1} + \frac{A2}{x - q2} + \ldots + \frac{An}{x - qn}$$

2. $Q(x) = (x - q1)^2 (x - q2)^3$.

$$f(x) = \frac{A}{x - q1} + \frac{B}{(x - q1)^2} + \frac{C}{x - q2} + \frac{D}{(x - q2)^2} + \frac{E}{(x - q2)^3}$$

Examples

$$(1).\int \frac{dx}{x^2-4x-5}$$

$$(2).\int \frac{x+1}{x^2-5x+6}dx$$

$$(3).\int \frac{2x^2+3}{x(x-1)^2}dx$$

$$(4).\int \frac{2x^2-2x-1}{x^3-x^2}dx$$

$$(5).\int \frac{(x^2+1)dx}{x^2-4x-5}$$

$$(6).\int \frac{x^3}{x^2-x-6}dx$$

$$(7).\int \frac{x^2}{(x-1)^3}dx$$

$$(8).\int \frac{2x^2+3x+3}{(x+1)^3}dx$$

The Non Linear Factor

1. $Q(x) = (x^2 + q1)(x^2 + px + q2)$

$$f(x) = \frac{A + Bx}{x^2 + q1} + \frac{C + Dx}{x^2 + px + q2}$$

2. $Q(x) = (x^2 + q1)^2 (x^2 + q2)$.

$$f(x) = \frac{A + Bx}{x^2 + q1} + \frac{C + Dx}{\left(x^2 + q1\right)^2} + \frac{E + Fx}{\left(x^2 + q2\right)}$$

Soal latihan

- Selesaikan Integral tak tentu berikut

$$\int \frac{dx}{\sqrt{16+x^2}} \qquad \int \frac{(x-2)}{x^2-4x+3}\,dx$$

Examples

$$(1).\int \frac{x^3+3x^2+x+9}{(x^2+1)(x^2+4)}dx$$

$$(2).\int \frac{dx}{x(x^2+x+1)}$$

$$(3).\int \frac{3x^2+12x+2}{(x^2+4)^2}dx$$

$$(4).\int \frac{4x+2}{x^4+2x^3+x^2}dx$$

Integrals Involving Rational Exponents

- The method can be used for calculating integrals with integrand has rational power of x
  - $x^{1/n}$, substitute $u = x^{1/n}$.
  - $x^{1/n}$ and $x^{1/m}$, substitute $u = x^{1/mn}$.
- Examples :

$$(1). \int x\sqrt{x-2}\,dx$$

$$(2). \int_4^8 \frac{\sqrt{x-4}}{x}\,dx$$

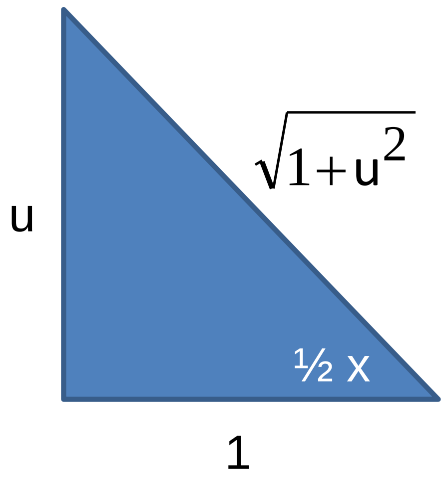
$$(3). \int \frac{dx}{\sqrt{x}+\sqrt[3]{x}}$$

Integrals involving Rational Expressions in sin x and cos x

- We will calculate the integral of f(x) in form of rational expressions in sin x and cos x
- Substitute :
  - $u = \tan \frac{1}{2} x$ and $du = \frac{1}{2} \sec^2 \frac{1}{2} x \, dx$, $-\pi < x < \pi$ and
  - $\sin x = \sin 2(\frac{1}{2} x) = 2 \sin \frac{1}{2} x \cos \frac{1}{2} x$
  - $\cos x = \cos 2(\frac{1}{2} x) = 1 - 2\sin^2 \frac{1}{2} x$

$$\sin x = \frac{2u}{1+u^2}$$

$$\cos x = \frac{1-u^2}{1+u^2}$$



Examples

$$(1).\int \frac{dx}{1+\sin x}$$

$$(2).\int_{\pi/2}^{\pi} \frac{dx}{1-\cos x}$$

$$(3).\int \frac{dx}{\sin x+\tan x}$$


Thank You!