Applications of Differentiation
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How to Sketch the graph of function

1. The X-intercept and Y-intercept
2. Intervals of monotonic and extreme points
3. Intervals of concavity and inflection points
4. Asymptotes

1. The X-intercept and Y-intercept

Assume y = f(x).

Point of intercept between the function and the axis, i.e:

X – axis if y = 0 and Y – axis if x = 0

Example :

Find the intercept point of function $f(x) = x^2 - 3x - 4$ with the axis

2. Intervals of monotonic and extreme points (1)

The interval of monotonic of $y = f(x)$:

1. Intervals of increase,

   f(x) is increasing on the interval if at any point x1 and x2, in this interval, we have f(x1) > f(x2) for x1 > x2

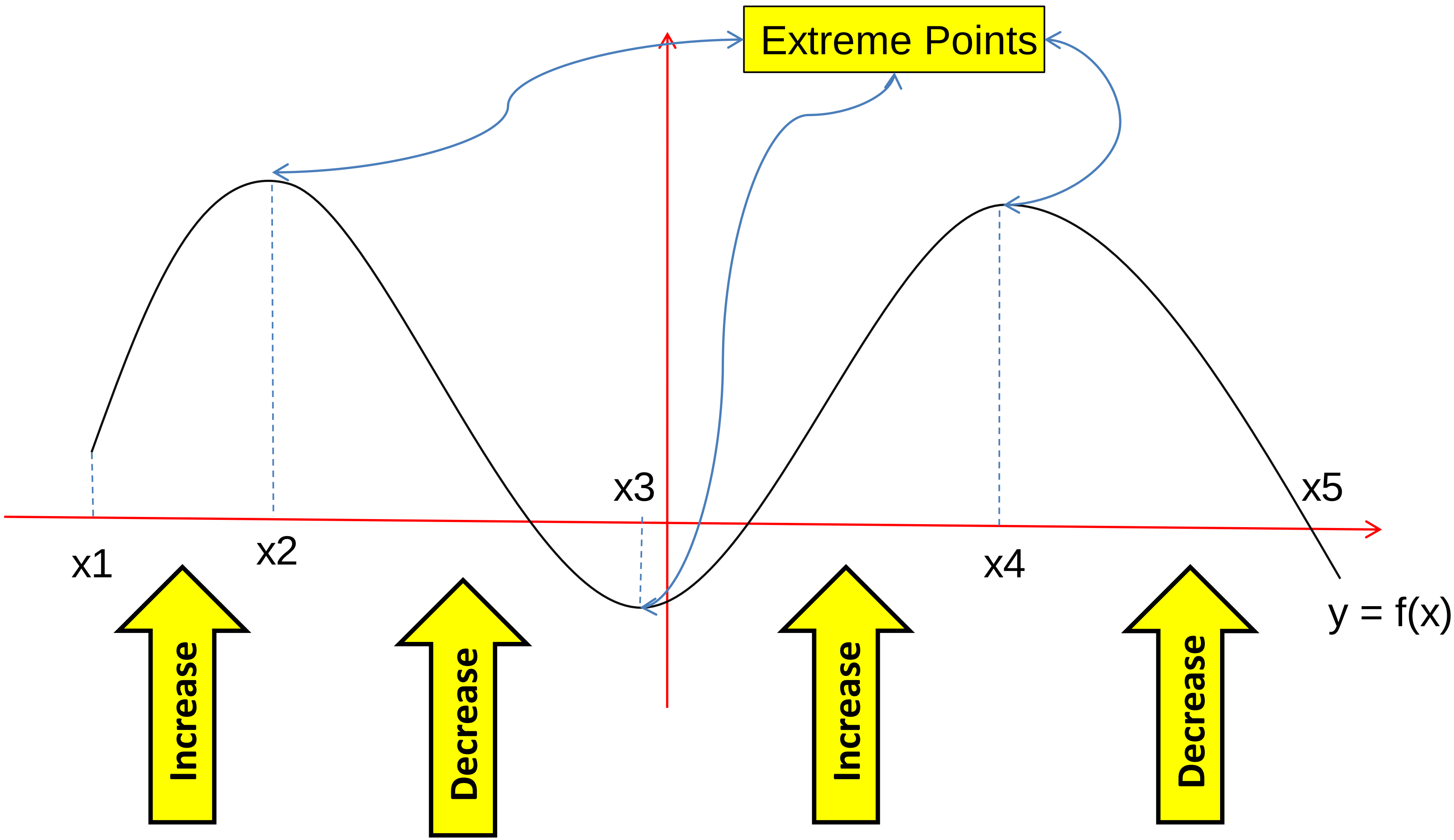
   if f '(x) > 0 on (a,b) then f is increasing on (a,b)

2. Intervals of decrease,

   f(x) is decreasing on the interval if at any point x1 and x2 in this interval , we have f(x1) > f(x2) for x1 < x2.

   if f '(x) < 0 on (a,b) then f is decreasing on (a,b)

2. Intervals of monotonic and extreme points (2)



Examples

Find the intervals of monotonic of this functions

1. $f(x) = x^2 - 5x + 6$
2. $f(x) = 5 + 12x - x^3$
3. $f(x) = x / (x^2 + 2)$
4. $f(x) = (x - 1) / (x - 2)$
5. $f(x) = 8 / (4 - x^2)$

2. Intervals of monotonic and extreme points (3)

(Relative) Extreme Points :

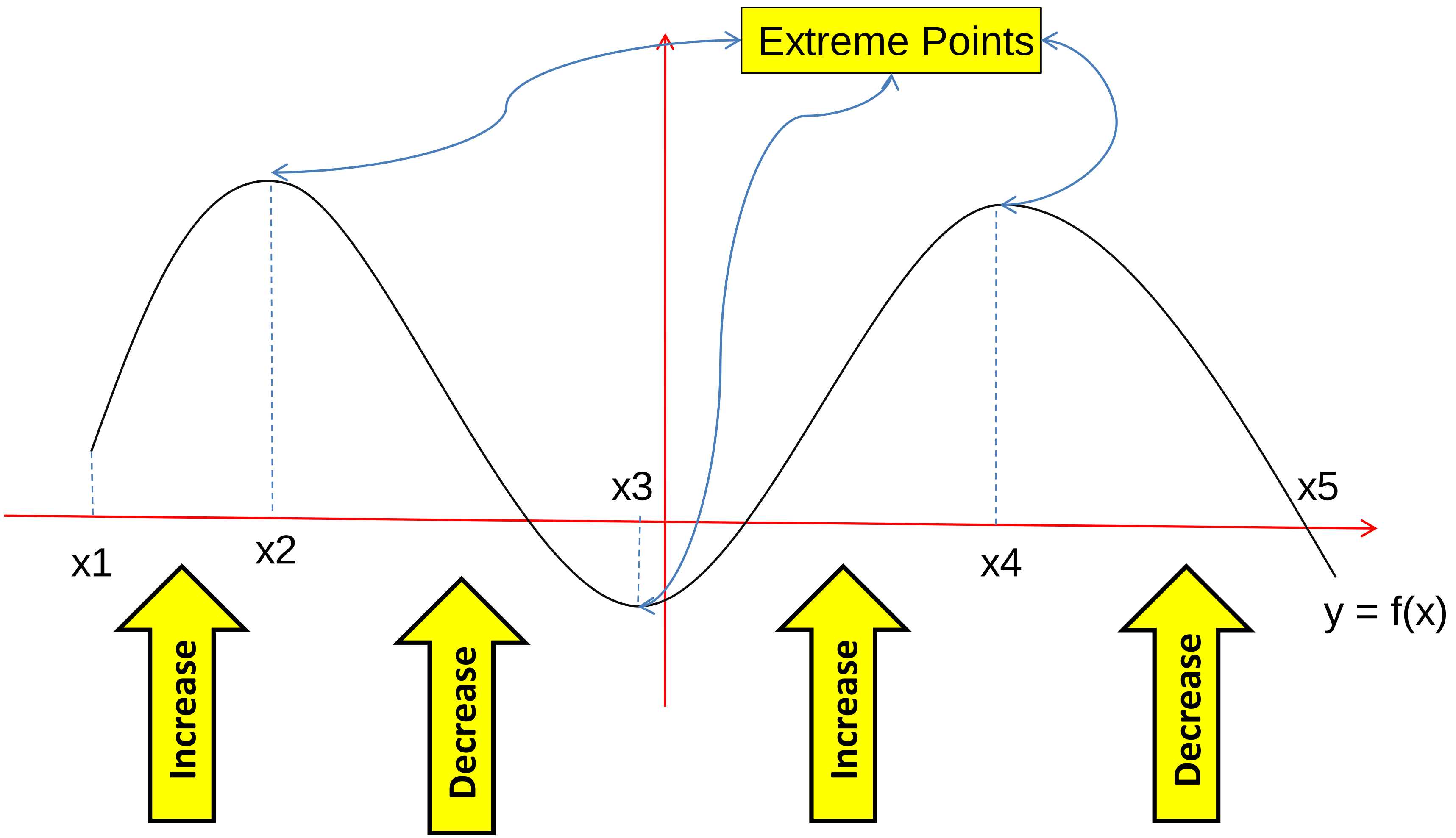
1. Maximum point, (x0,f(x0))

   f have (relative) maximum at x0 if f(x0) ≥ f(x) for all x in some interval containing x0

2. Minimum point, (x0, f(x0))

   f have (relative) minimum at x0 if f(x0) ≤ f(x) for all x in some interval containing x0

2. Intervals of monotonic and extreme points (2)



2. Intervals of monotonic and extreme points (4)

(Relative) Extreme Points of y = f(x) can be found by two methods i.e : (Assume xo such that f'(x0) = 0)

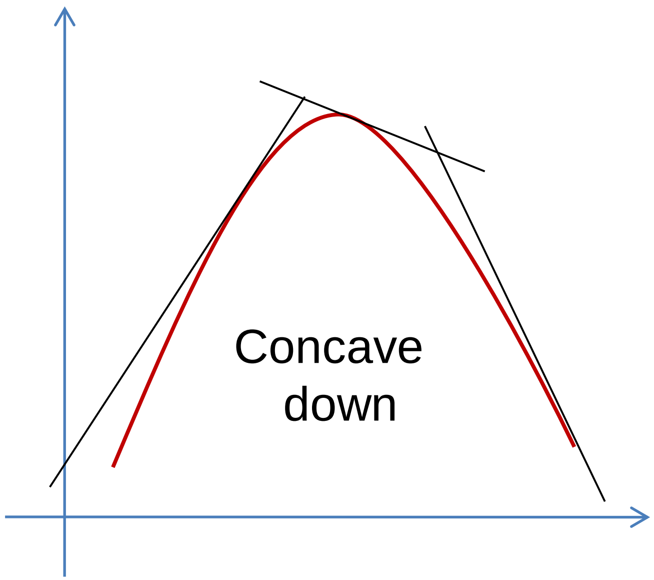
1. The first derivative,
   – if f'(xo) > o on an interval extending left from xo and f'(x0) < 0 on an interval extending right from xo then f has (relative) maximum at xo
   – if f'(xo) < o on an interval extending left from xo and f'(x0) > 0 on an interval extending right from xo then f has (relative) minimum at xo
2. The second derivative
   – If f"(x0) > 0 then f has (relative) minimum at xo
   – If f"(x0) < 0 then f has (relative) maximum at x0

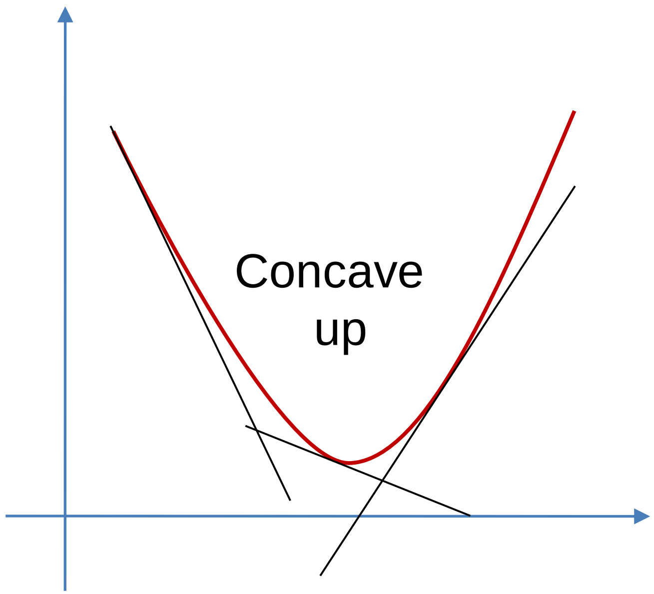
Examples

Find the extreme point and classify them as maximum point or minimum point

1. $f(x) = x^2 - 5x + 6$
2. $f(x) = 5 + 12x - x^3$
3. $f(x) = 3x^4 - 4x^3$
4. $f(x) = x(x + 2)^2$
5. $f(x) = (x^2 - 3) / (x^2 + 1)$
6. $f(x) = x^2 / (1 + x^2)$

3. Interval of Concavity (1)





- The curve lies below its tangent lines
- If we travel left to right along this curve so the slope (gradient) of tangent line decrease

- The curve lies above its tangent lines
- If we travel left to right along this curve so the slope (gradient) of tangent line increase

3. Intervals of concavity (2)

Let f is differentiable on interval I

a. f is called concave up on interval I if $f'(x)$ is increasing on interval I

b. f is called concave down on interval I if $f'(x)$ is decreasing on interval I

Second derivative test for concavity

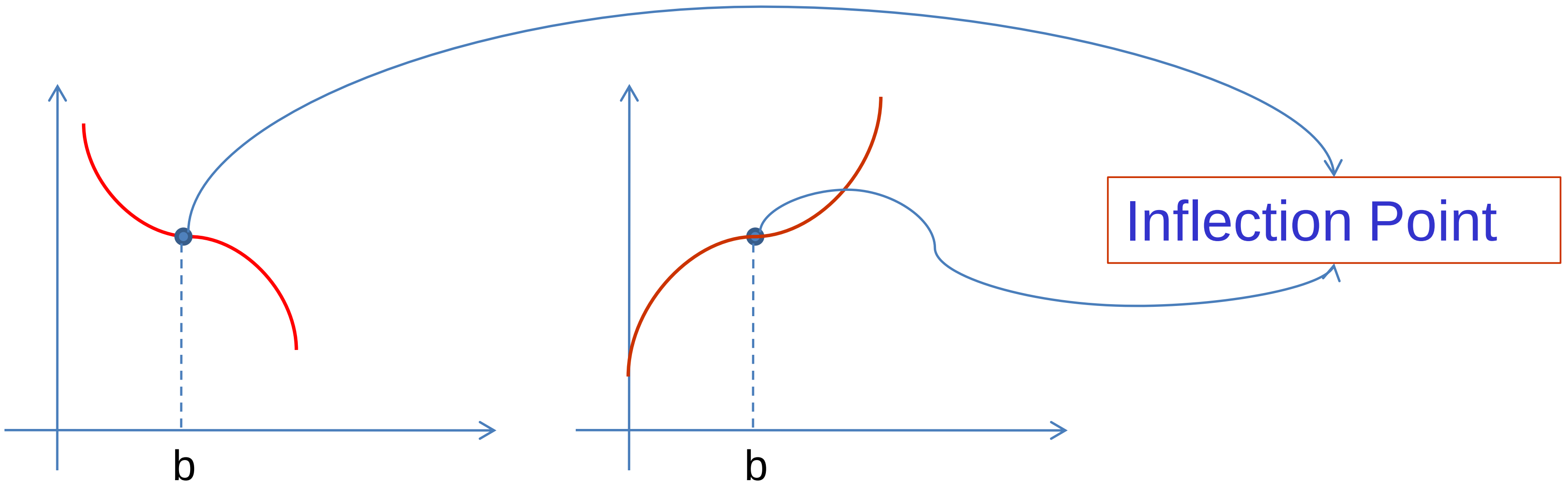
1. If $f''(x) > 0$ on interval I then f is concave up on I.
2. If $f''(x) < 0$ on interval I then f is concave down on I .

Problems

Find the interval concavity of

1. $f(x) = 2x^5 - 15x^4 + 30x^3 - 6$
2. $f(x) = \dfrac{x^2 - 3x + 1}{x - 3}$
3. $f(x) = \dfrac{x^2 - 2x + 1}{x - 2}$
4. $f(x) = \dfrac{(x+1)^2}{x}$
5. $f(x) = x^{1/3}$

4. Inflection Point (1)



Let f is continuous on an open interval containing x = b.

If f changes the direction of its concavity at x = b, then point (b,f(b)) on the graph of f is called an inflection point of f.

4. Inflection points (2)

Theorem : The function of f(x) has an inflection point at x = b if :

- f(x) has the second derivative at x = b such that f”(b) = 0
- f(x) has not the second derivative at x = b or f”(b) is not defined

4. Inflection points (3)

How to find the inflection Point of f(x) :

- Find x = b such that f "(b) = 0 or f"(b) is not defined.
- does f(x) change the direction of its concavity at x = b? If f(b) is defined and f(x) changes the direction of its concavity at x = b then (b, f(b)) is inflection point of f(x).

Example : find the inflection point of

1. $f(x) = x^3$
2. $f(x) = x^4 - 1$
3. $f(x) = x^{1/3} - 2$

Problems

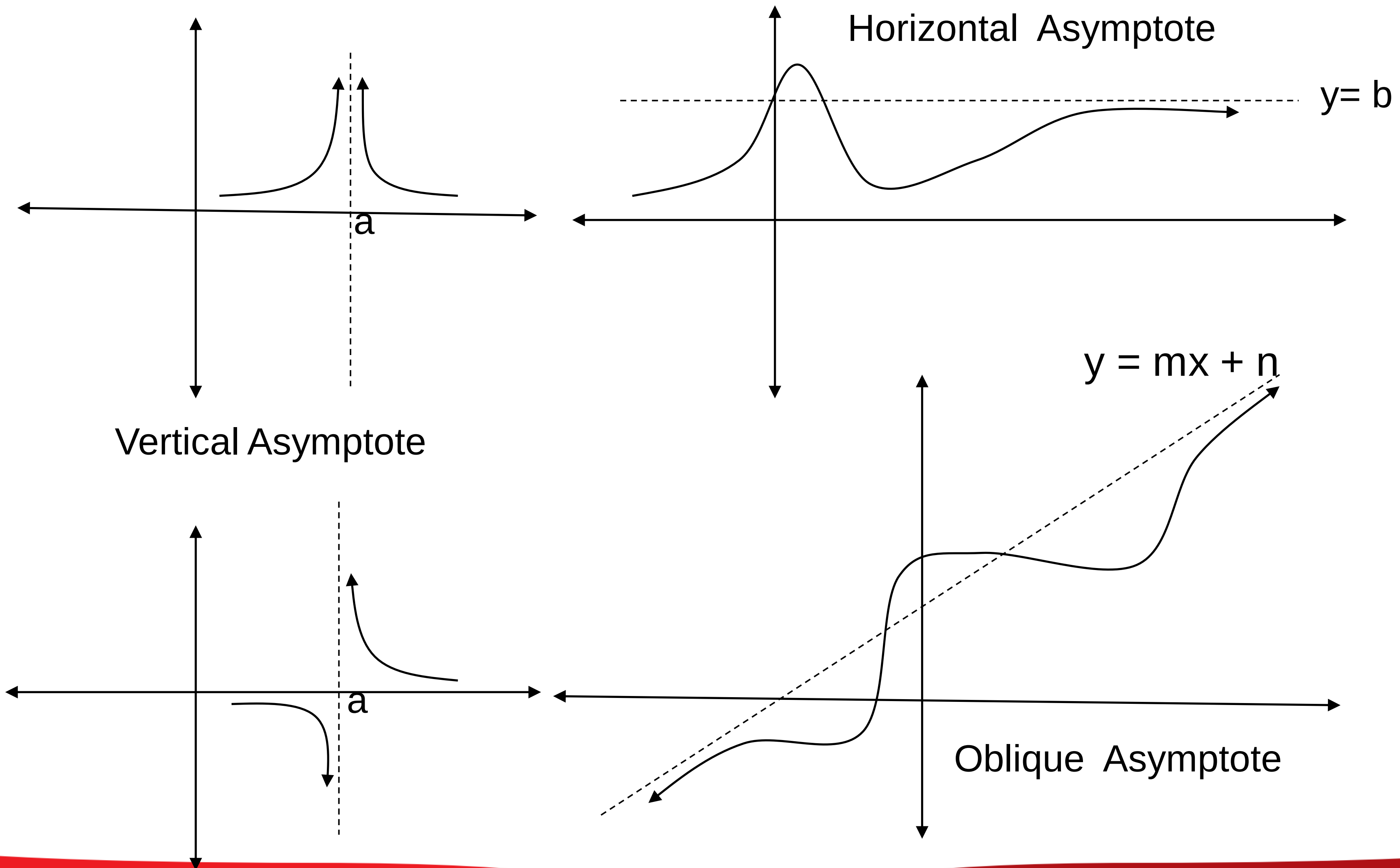
Find the inflection points of

1. $f(x) = (x + 2)^3$
2. $f(x) = x^4 – 8x^2 + 16$
3. $f(x) = x / (x^2 + 2)$
4. $f(x) = 3x^4 – 4x^3$
5. $f(x) = x^{4/3} – x^{1/3}$
6. $f(x) = x^{1/3}(x+4)$

5. Asymptote (1)

- Definition : the line is said as an asymptote of f(x) if the curve of f(x) tends toward this line.
- The kind of asymptote
  1. Vertical asymptote ($x = a$)
  2. Horizontal asymptote ($y = b$)
  3. Oblique asymptote ($y = mx + n$)

5. Asymptote (2)



5. Asymptote (3)

- The rational function, $f(x) = p(x)/q(x)$ has an asymptote
- A line $x = a$ is called a vertical asymptote for the graph of $f(x)$ if $f(x) \to \pm\infty$ as $x \to a$ (from the left or the right)
- A line $y = b$ is called a horizontal asymptote for the graph of $f(x)$ if $f(x) \to b$ as $x \to \pm\infty$

Vertical Asymptote

- Let $f(x) = p(x) / q(x)$
- The vertical asymptote of $f(x)$ can be found from the x-intercepts of $q(x)$
- Examples :
  1. $f(x) = 2x / (x- 3)$
  2. $f(x) = (x – 1) / (x^2 – 4)$
  3. $f(x) = 2 + 3/x – 1/x^3$
  4. $f(x) = (x^2 – 1)/(x^2 – 2x – 3)$

Horizontal Asymptote

- Let f(x) = p(x) / q(x)
- The horizontal asymptote of f(x) can be found from computing of $\lim_{x \to \infty} f(x)$ and $\lim_{x \to -\infty} f(x)$

  If either limit has the finite value b then the line y = b is a horizontal asymptote
- Examples :
  1. f(x) = 2x / (x- 3)
  2. $f(x) = 2 + 3/x - 1/x^3$
  3. $f(x) = \dfrac{2x}{\sqrt{9x^2 - 1}}$

Oblique Asymptote

- If $f(x) = p(x) / q(x)$ is such that the degree of the numerator exceeds the degree of the denominator by one, then the graph of $f(x)$ has an oblique asymptote
- Examples :
  1. $f(x) = (x^2 - 1) / (x + 2)$
  2. $f(x) = x^3 / (x^2 – 2x – 3)$
- This rational function can be written as :

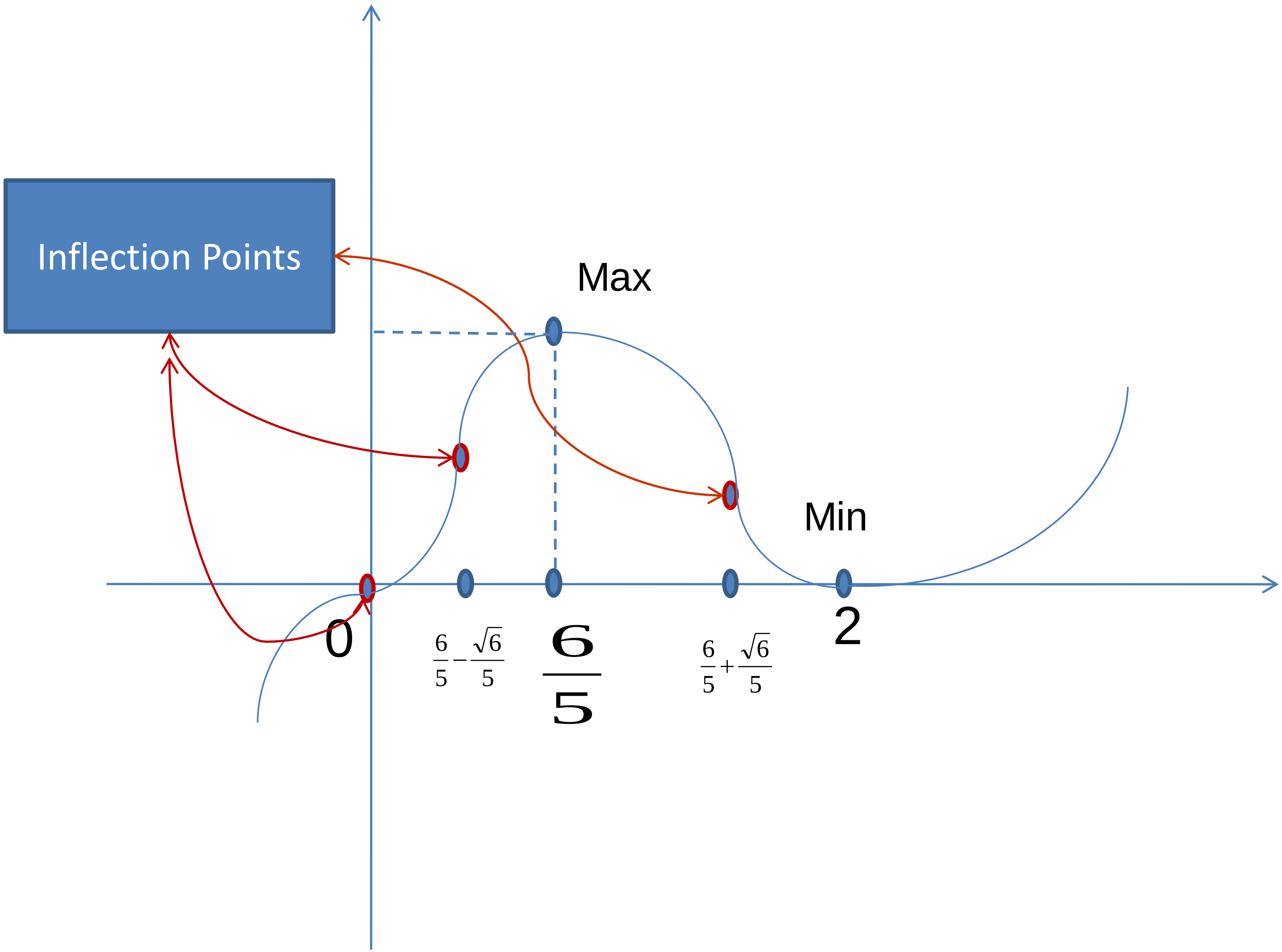
  $f(x) = (mx + n) + r(x)/q(x)$ and $y = mx + n$ is an oblique asymptote
- if the rational function has an oblique asymptote then it has not a horizontal asymptote and its contrary

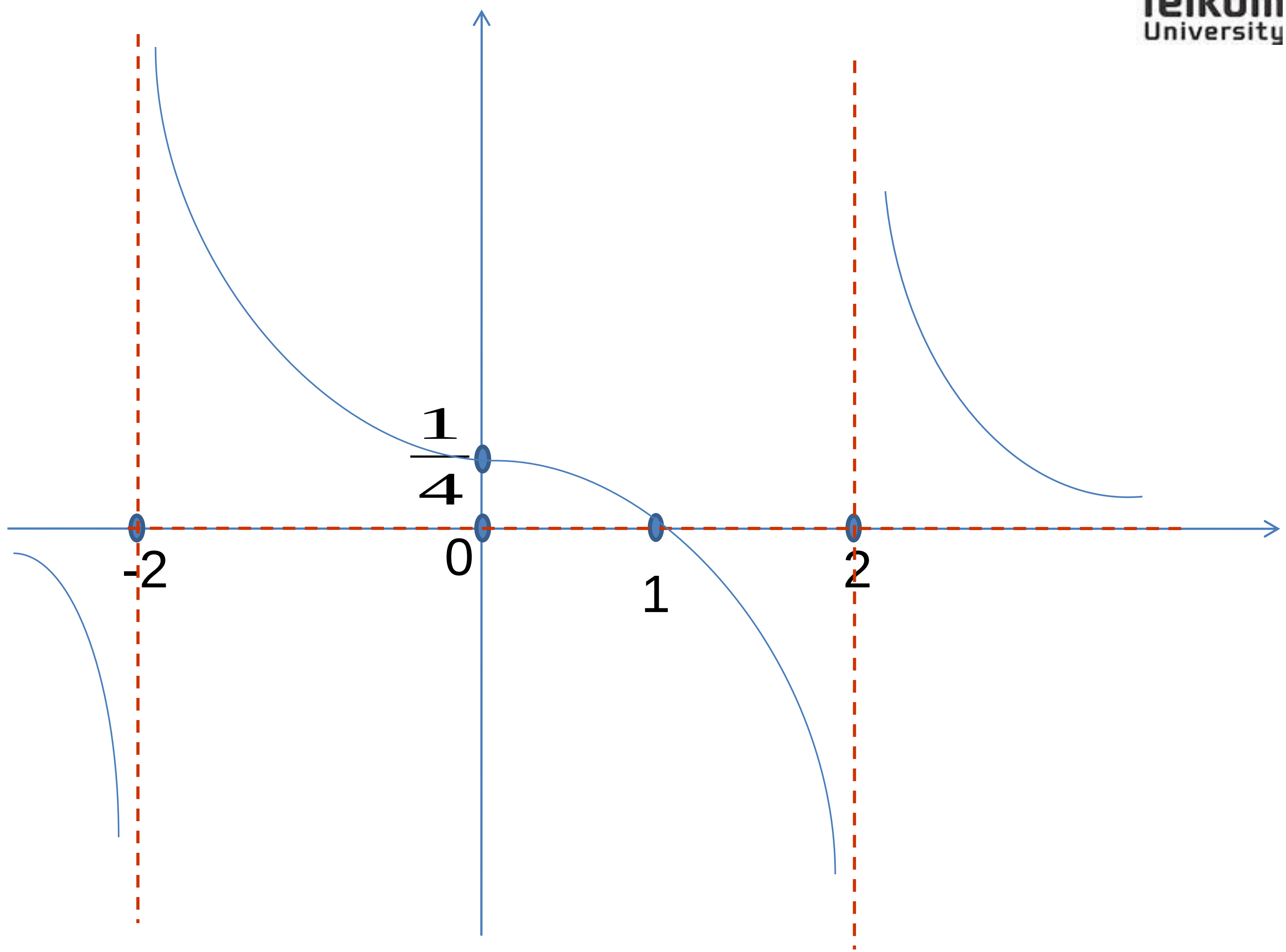
The graph of the function

- Let f(x) is polynomial function, $f(x) = a_0 + a_1x + a_2x^2 + \dots$
- If we will sketch this graph then we must find
  1. The x-intercept of f(x) and y-intercept of f(x)
  2. The interval of monotonic and extreme points
  3. The interval of concavity and inflection points
- If we will sketch the graph of the rational function, f(x) = p(x)/q(x) then we must find the asymptote (vertical, horizontal or oblique)

Problems

- Sketch the graph of this polynomial functions :
  1. $f(x) = x^3 + 3x^2 + 5$
  2. $f(x) = x^4 – 2x^2 – 12$
  3. $f(x) = x^5 – 4x^4 + 4x^3$
- Sketch the graph of rational functions :
  4. $f(x) = (1 – x) / x^2$
  5. $f(x) = (x – 1)/(x^2 – 4)$
  6. $f(x) = (x^2 – 2x – 3)/(x + 2)$


Inflection Points
Max
Min
0
$\frac{6}{5}-\frac{\sqrt{6}}{5}$
$\frac{6}{5}$
$\frac{6}{5}+\frac{\sqrt{6}}{5}$
2



1/4
-2
0
1
2

L'hopital Rule (1)

- Indeterminate forms of limit are $\frac{0}{0}, \frac{\infty}{\infty}, 0.\infty, \infty - \infty$
- This forms can be solved by L'hopital Rule

1) Let lim f(x) = lim g(x) = 0. Then lim f(x)/g(x) = lim f'(x)/g'(x)
2) Let lim f(x) = lim g(x) = ∞. Then lim f(x)/g(x) = lim f'(x)/g'(x)
3) Let lim f(x) = 0 and lim g(x) = ∞. Then lim f(x)g(x) can be written as lim f(x)g(x) = lim f(x)/[1/g(x)] with the form 0/0 or lim f(x)g(x) = lim g(x)/[1/f(x)] with the form ∞/∞.
4) Let lim f(x) = ∞ and lim g(x) = ∞. Then lim [f(x) – g(x)] can be rearranged into the form 0/0 or ∞/∞

L'hopital Rule (2)

Find the limits

1. $\lim\limits_{x \to 0} \dfrac{x - \sin x}{2 - 2\cos x}$

2. $\lim\limits_{x \to +\infty} \dfrac{2x + 1}{2 - 5x}$

3. $\lim\limits_{x \to 0} 2x \csc x$

4. $\lim\limits_{x \to 0} \cot 2x \left(1 - \cos 2x\right)$

5. $\lim\limits_{x \to +\infty} \left(\sqrt{x^2 + x} - x\right)$

6. $\lim\limits_{x \to -\infty} \left(\sqrt{x^2 - 3x} - \sqrt{x^2 - 3}\right)$

Solving Some applied optimization problems

- Step # 1 : draw an appropriate figure and label the quantities relevant to the problem
- Step # 2 : find the formula for the quantity to be maximized or minimized
- Step # 3 : express the quantity as a function of one variable
- Step # 4 : find the interval of monotonic to obtain the maximum or minimum.

Problems

1. Express the number 100 as a sum of two nonnegative terms whose product is as large as possible
2. How should two nonnegative numbers be chosen so that their sum is 1 and the sum of their squares is as small as possible ?
3. Find the dimensions of the rectangle with maximum area that can be inscribed in a circle of radius 10 cm
4. A rectangle has its two lower corner on the x-axis and its two upper corner on the curve $y = 16 - x^2$. For all such rectangles, what are the dimension of the one with largest area ?


Thank You!